Discretization Methods of Fractional Parallel PID Controllers

Mohamad Adnan Al-Alaoui*, *Senior Member, IEEE*
Electrical and Computer Engineering department
American University of Beirut
Lebanon

***Abstract*— This work addresses employing direct and indirect discretization methods to obtain a rational discrete approximation of continuous time parallel fractional PID controllers. The different approaches are illustrated by implementing them on an example.**

Index Terms-
Al-Alaoui operator, analog_to digital_conversion, bilinear transformation, fractional order systems, PID control, transfer functions.

I. Introduction

Fractional systems are systems that are represented by differential equations that allow non integer orders. This is a generalization of the integer order integration and differentiation. The order can take on any real value, not necessarily only fractional values. Thus the fractional designation is not accurate, however the misnomer is now the accepted designation. The corresponding transfer functions for fractional order systems resulting from applying continuous or discrete time transforms will be non-rational. An analytical solution for the simulation and computation of a fractional model's output is often not simple to obtain. Algorithms were developed to approximate the outputs of fractional systems by using either continuous or discrete rational models [5-9], [15], [17].

Indirect and direct discretization approaches to the time domain simulations of the output of a fractional system are the two major ones. The indirect approach first develops a rational analog model approximation of the fractional system then applies an s-to-z transform to the continuous time approximation. While, the direct approach first obtains a fractional discrete system from the fractional analog system by applying an s-to-z transform to the fractional analog system, then a rational discrete approximation of the fractional discrete system is obtained. The method of continued fraction expansion (CFE) is one of the early methods that was employed to obtain a rational analog or a rational discrete approximation of a fractional analog or a fractional discrete system respectively [13]. The versatile S. C. Dutta Roy had employed it to obtain rational continuous-time approximations to fractional analog systems[10-11]. The mathematical basis of the CFE are obtained in the classical works of Wall [16] and Khovanskii [12]. Other proposed continuous-time approximations were introduced by Oustaloup et. al.[15], and recently by Aoun et.al. [6] Xue et.al. [17]. The method of CFE was applied by Chen et. al. in a direct discretization approach [8-9]. Chen and Moore's work gave impetus to research employing CFE and the Al-Alaoui operator [8]. In the sequel, the step response of the unity feedback system of Figure 1, with a given fractional order parallel PID (proportional, integral, and derivative) controller, C(s), and an integer order plant, G(s), is determined [7]. Different discretization methods for C(s) are employed while G(s) is discretized by using the bilinear transform. In the following s is approximated by using the bilinear transform [14], the Al-Alaoui transform [1-3], and the phase enhanced Al-Alaoui transform [4], represented below by equations (1), (2), and (3) respectively. The phase enhanced Al-Alaoui transform is obtained by a half-sample advance of the Al-Alaoui transform, which is achieved by multiplying (2) by $z^{0.5}$, or dividing it by $z^{-0.5}$. The half-sample advance, shown in (3), changes the almost linear phase of (2) to almost 90 degrees while the magnitude remains the same.

$$s \approx \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})} \tag{1}$$

$$s \approx \frac{8}{7T}\frac{(1-z^{-1})}{(1+z^{-1}/7)} \tag{2}$$

$$s \approx \frac{8}{7T}\frac{z^{0.5}(1-z^{-1})}{(1+\frac{z^{-1}}{7})} = \frac{8}{7T}\frac{(1-z^{-1})}{z^{-0.5}(1+\frac{z^{-1}}{7})} \tag{3}$$

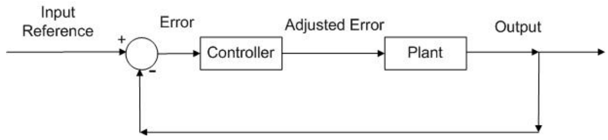

Figure 1. A Unity Feedback System

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II. PID CONTROL

PID controller is often employed as shown in Figure 1 to improve the performance of the unity feedback loop shown [7]. Two popular implementations of the PID controller are the parallel and the series implementations. The transfer function of the parallel implementation of the PID controller is expressed as

$$K_p + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s} \quad (4)$$

$K_p, K_I, and\ K_D$ designate the Proportional, Integral, and Derivative gains respectively with each one serving different purposes like reducing rise time or steady state error, etc...

For the $PI^{\lambda}D^{\mu}$ the transfer function of the parallel implementation is expressed as follows:

$$K_p + \frac{K_I}{s^{\lambda}} + K_D s^{\mu} = \frac{K_D s^{\lambda+\mu} + K_P s^{\lambda} + K_I}{s^{\lambda}} \quad (5)$$

The fractional $PI^{\lambda}D^{\mu}$ controller has five degrees of freedom,($K_p, K_I, K_D, \lambda, and\ \mu$), versus three degrees of freedom for the integer PID controller. The series $PI^{\lambda}D^{\mu}$ controller is implemented as a cascade of $PI^{\lambda}(s)$ and $PD^{\mu}(s)$. In this work the parallel implementation of Cao et. al. [7] will be addressed. The transfer functions of controller C(s), and the plant G(s) are presented below as equations (6) and (7).

$$C(s) = 2.7566 + 0.0029 s^{-0.7908} + s^{0.4848} \quad (6)$$

$$G(s) = \frac{400}{s^2 + 50s} \quad (7)$$

III. INDIRECT AND DIRECT DISCRETIZATION METHODS

In this section direct and indirect discretization approaches are carried out to obtain rational discrete approximations of C(s) and the different Bode plots of the approximations are compared to the ideal response. Additionally G(s) is discretized using the bilinear transform and the resulting step response of the closed system with various C(s) discretizations are compared.

A. Indirect Discretization

Indirect discretization consists of two steps. In the first step a rational analog transfer function that approximates the irrational transfer function of the fractional controller is obtained. In the second step an analog to digital transformation of the rational analog approximation is done.

1) The Analog Rational Approximation

The approach described in [6] and [17] will be implemented. The approximation is carried out in the following two steps.

- Determine a frequency range [ω_A, ω_B] where we need to approximate the fractional order transfer function.
- Perform an analog integer-order approximation of the fractional-order transfer function

This is accomplished in [6] by letting:

$$s^{-r}_{[\omega_A,\omega_B]} \cong \frac{\omega_B(\omega_A + s)}{(\omega_A)^n\left(rs^2 + \omega_B s + (1-r)\omega_A\omega_B\right)}\left(\frac{1+\frac{s}{\omega_B}}{1+\frac{s}{\omega_A}}\right)^r \quad (8)$$

Note that (8) is equation (22) in [6].

Then equation (8) is expanded using CFE, Taylor series expansion, or the method of recursive poles and zeros to obtain an analog approximation. The analog approximation with recursive poles and zeros is described in detail in [17], and will be used in this work. It is summarized below:

$$\left(\frac{1+\frac{s}{\frac{d}{b}\omega_A}}{1+\frac{s}{\frac{b}{d}\omega_B}}\right)^r = \lim_{N\to\infty}\prod_{k=-N}^{k=N}\frac{1+s/\omega'_k}{1+s/\omega_k} \quad (9)$$

$$\omega'_k = \left(\frac{d}{b}\omega_A\right)^{\frac{r-2k}{2N+1}} \quad (10)$$

$$\omega_k = \left(\frac{b}{d}\omega_B\right)^{\frac{r+2k}{2N+1}} \quad (11)$$

$$s^r \approx K\left(\frac{ds^2 + bs\omega_B}{d(1-r)s^2 + bs\omega_B + dr}\right)\prod_{k=-N}^{k=N}\frac{s+\omega'_k}{s+\omega_k} \quad (12)$$

$$K = \left(\frac{d}{b}\omega_A\right)^r\prod_{k=-N}^{k=N}\frac{\omega_k}{\omega'_k} \quad (13)$$

We used b = 10 and d = 9 as in [17].

The procedure for the approximation can be briefly summarized in the following steps.

a) The frequency range [ω_A, ω_B] and N are given.
b) Based on the fractional order r, calculate ω'_k and ω_k according to (10) and (13)
c) Compute K from (13)
d) Obtain the approximate rational transfer function from (12) to replace s^r

The following values are used:
b = 10; d = 9; ω_A = 0.01; ω_B = 100; N = 3;
r1 = -0.7908; r2 = 0.4848; % Fractional exponents

The above analog approximation yields the following transfer function.

1629 s^{18} + 4.776e005 s^{17} + 5.253e007 s^{16} + 2.815e009 s^{15} + 7.908e010 s^{14} + 1.179e012 s^{13} + 9.335e012 s^{12} + 3.9e013 s^{11} + 8.557e013 s^{10} + 9.798e013 s^9 + 5.81e013 s^8 + 1.77e013 s^7 + 2.722e012 s^6 + 2.065e011 s^5 + 7.255e009 s^4 + 1.003e008 s^3 + 1.128e005 s^2 - 5606 s - 20.25

74.73 s^{18} + 3.205e004 s^{17} + 4.701e006 s^{16} + 3.196e008 s^{15} + 1.103e010 s^{14} + 1.979e011 s^{13} + 1.836e012 s^{12} + 8.794e012 s^{11} + 2.154e013 s^{10} + 2.698e013 s^9 + 1.714e013 s^8 +

$$5.509e012\ s^7 + 8.816e011\ s^6 + 6.886e010\ s^5 + 2.465e009\ s^4 + 3.434e007\ s^3 + 3.207e004\ s^2 - 2017\ s - 7.347 \quad (14)$$

2) Discretizing the Analog Approximation

To discretize the analog rational approximation of (14) select suitable s-to-z transforms. The bilinear transform, equation (1), Al-Alaoui transform, equation (2), and Al-Alaoui-1 transform, equation (3), are employed. Replacing s in (14) by the right hand side of equations (1) or (2) yield two rational discrete time transfer functions, while replacing s in (14) by the right hand side of equation (3) yields an 'almost' rational discrete time function, a discrete time function with additional delay multipliers. The resulting three discrete time functions are plotted in Figure 2. Concerning Al-Alaoui-1 indirect method resulting from substituting (3) in (14), the transfer function can be written in the following form: $[N_1(z)+z^{0.5}(N_2(z)]/[D_1(z)+z^{0.5}D_2(z)]$, where $N_1(z), N_2(z)$, $D_1(z)$ and $D_2(z)$ are rational polynomials. This is due to the fact that when s has an even power this will yield $N_1(z)$ in the numerator and $D_1(z)$ in the denominator. Whereas when the power is odd then $z^{0.5}$ can be factored outside yielding $N_2(z)$ in the numerator and $D_2(z)$ in the denominator. An alternative to the above approach is to substitute for the half sample advance or delay the approximations proposed in [4].

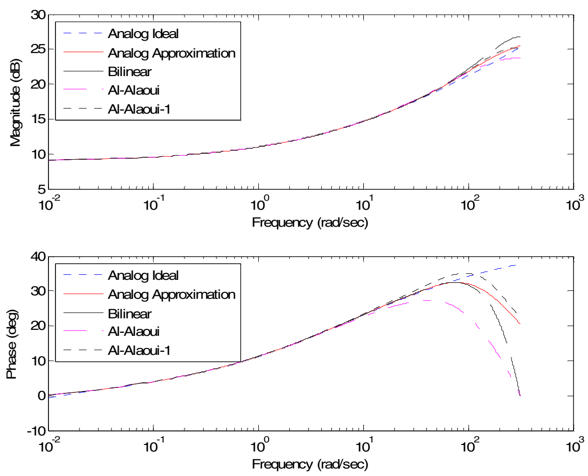

Figure 2. The frequency response of the ideal, the analog approximation, the discretization of the analog approximation using the bilinear , the Al-Alaoui and the Al-Alaoui-1 transformations.

IV. Direct Discretization

The direct discretization approach consists of two steps. In the first step an s-to-z transform is applied to the irrational transfer function of the fractional order controller to obtain an irrational discrete time transfer function, a generating function, of the controller. The direct discretization in this paper applies the CFE method to the generating function, to obtain digital rational approximations, IIR filters, to the continuous fractional order controller C(s) [5], [7-9].

A. Generating Functions

Direct discretization of s^r can be expressed by the generating function obtained from an s-to z transform $s = D_i(z)$, where i =1, 2, or 3 corresponding to equations (1)-(3) and $D_i(z)$ correspond to the right hand side of equations (1)-(3). Then apply CFE to approximate the fractional-order generating function, $G_i(z) = (D_i(z))^r$, by a rational disctret time , IIR, transfer function [5], [8-9]

B. CFE

The second step obtains a rational discrete transfer function of the controller that approximates the irrational discrete time function that was obtained in the first step. The second step could employ a variety of methods such power series expansions, or CFE. Replace s by (1) - (3), then use the CFE to approximate the obtained digital fractional-order transfer function.

We let: `NT = 50 (Order of Taylor Series Expansion);m = 9; n = 9; T = 0.01;`

1) The CFE of the bilinear discretization

Applying the bilinear transform to equation (6) then applying the CFE approximation method yields the following rational transfer function, if we only considered it up to its 18th order.

$$\frac{15.8 z^{18} - 17.49 z^{17} - 56.59 z^{16} + 64.15 z^{15} + 81.94 z^{14} - 96.07 z^{13} - 61.36 z^{12} + 75.58 z^{11} + 25.15 z^{10} - 33.47 z^9 - 5.439 z^8 + 8.297 z^7 + 0.5236 z^6 - 1.073 z^5 - 0.008252 z^4 + 0.06111 z^3 - 0.0008988 z^2 - 0.00105 z + 5.231e\text{-}005}{z^{18} - 0.306 z^{17} - 4.214 z^{16} + 1.197 z^{15} + 7.369 z^{14} - 1.917 z^{13} - 6.918 z^{12} + 1.619 z^{11} + 3.766 z^{10} - 0.7735 z^9 - 1.197 z^8 + 0.2078 z^7 + 0.2124 z^6 - 0.02933 z^5 - 0.01873 z^4 + 0.001838 z^3 + 0.0006293 z^2 - 3.267e\text{-}005 z - 5.083e\text{-}006} \quad (15)$$

2) The CFE of Al-Alaoui and Al-Alaoui-1 discretizations

Applying the Al-Alaoui transform, equation (2), to equation (6) then applying CFE method to the resulting irrational disctrete time function yields the following rational transfer function if we only considered it up to its 18th order.

$$\frac{12.7 z^{18} - 105.7 z^{17} + 398.4 z^{16} - 897.1 z^{15} + 1343 z^{14} - 1403 z^{13} + 1044 z^{12} - 553.3 z^{11} + 203.3 z^{10} - 47.97 z^9 + 5.545 z^8 + 0.318 z^7 - 0.1912 z^6 + 0.01836 z^5 + 0.0007462 z^4 - 0.000199 z^3 + 4.167e\text{-}006 z^2 + 3.157e\text{-}007 z + 1.669e\text{-}009}{z^{18} - 7.889 z^{17} + 28 z^{16} - 58.94 z^{15} + 81.62 z^{14} - 77.79 z^{13} + 51.74 z^{12} - 23.66 z^{11} + 6.994 z^{10} - 1.077 z^9 - 0.02814 z^8 + 0.0433 z^7 - 0.005877 z^6 - 0.0001575 z^5 + 8.696e\text{-}005 z^4 - 2.923e\text{-}006 z^3 - 3.441e\text{-}007 z^2 + 1.238e\text{-}008 z + 8.385e\text{-}011} \quad (16)$$

The CFE corresponding to Al-Alaoui-1 transform is obtained, as in Chen and Moore [8], by applying phase compensation to (16). Figure 3 plots the corresponding frequency responses of the direct discretization approach employing CFE.

V. Conclusions

Figures 2-4 demonstrate that the Al-Alaoui and the Al-Alaoui-1 transforms approach the ideal magnitude response better than the bilinear transform at high frequencies.

However, the bilinear transform phase is an ideal 90 degrees while the Al-Alaoui's presents an almost linear phase starting from 90 degrees at low frequencies with that disadvantage overcome by Al-Alaoui-1. Figure 2 presents the results of the indirect discretization approach of controller, C(s) and reflects clearly the properties of the transforms used. Figure 3 presents the results of the direct discretization of C(s) using the different transforms and CFE. Figure 4 shows that the lowest absolute magnitude error is obtained by using direct discretization using Al-Alaoui+CFE. On the other hand it is observed that all the responses of the indirect discretization methods perform better than the direct discretization approach using bilinear + CFE. If we consider the step responses of the unity feedback system using the analog approximation and the responses due to direct discretization it should be known that direct discretization methods yield shorter rise time than the analog approximation which the indirect approaches can at best approximate.

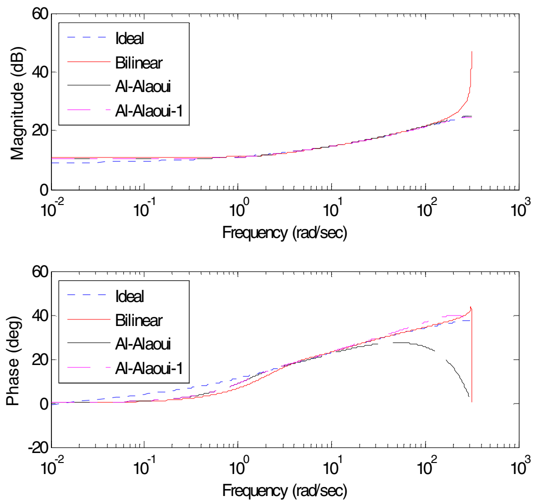

Figure 3. CFE comparison results for bilinear and Al-Alaoui discretizations.

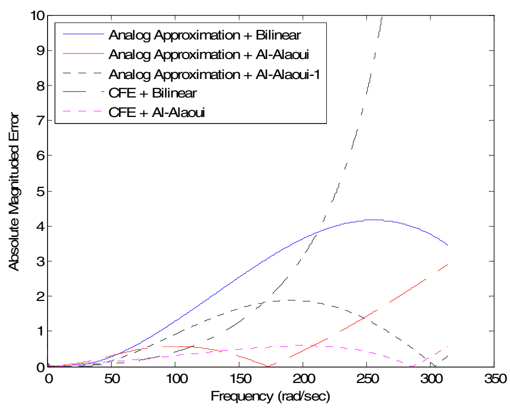

Figure 4. Error Comparison of Direct and Indirect Discretization Methods.

The above shows that there is no inherent reason to think that any of the two discretization approaches, direct or indirect, is superior to the other in terms of accuracy. This will depend on the different approximations that evolve over time for both approaches in addition to the problem at hand.

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